ON THE PROBLEM OF MODELING FOR PARAMETER IDENTIFICATION IN DISTRIBUTED STRUCTURES

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Abstract

Structures are often characterized by parameters, such as mass and stiffness, that are spatially distributed. Parameter identification of distributed structures is subject to many of the difficulties involved in the modeling problem, and the choice of the model can greatly affect the results of the parameter identification process. Analogously to control spillover in the control of distributed-parameter systems, identification spillover is shown to exist as well and its effect is to degrade the parameter estimates. Moreover, as in modeling by the Rayleigh-Ritz method, it is shown that, for a Rayleigh-Ritz type identification algorithm, an inclusion principle exists in the identification of distributed-parameter systems as well, so that the identified natural frequencies approach the actual natural frequencies monotonically from above.

Introduction

Structures are distributed-parameter systems, which implies that they are characterized by parameters, such as mass and stiffness, that are spatially distributed. The motion of distributed-parameter structures is governed by partial differental equations (pde's), in which the parameters appear in the form of space-dependent coefficient functions. For the most part, these pde's do not admit closed-form solutions, so that one must be content with approximate solutions. To

obtain an approximate solution, it is necessary to resort to spatial discretization, which amounts to representing the motion of the structure by a linear combination of admissible functions depending on the spatial variables multiplied by time-dependent generalized coordinates (Ref. 1). Certain integrations over the structure eliminate the dependence on the spatial variables. The net result is the transformation of the pde's into a set of ordinary differential equations (ode's), in which the parameters enter into the coefficient matrices. The process of obtaining the set of ode's is equivalent to deriving a discretized model designed to approximate the distributed model of the actual structure. How well the discrete model is capable of representing the distributed model depends on the number and type of admissible functions used in the discretization process. Indeed, the rate of convergence to the exact solution depends on the number of admissible functions and, perhaps to a larger extent, on the type of admissible functions used (Ref. 2). The discretized model can be used to compute a finite number of lower modes of vibration. It is symptomatic of such approximations that an even smaller number of computed lower natural frequencies than the number of degrees of freedom of the discrete model are accurate representations of the actual natural frequencies (Ref. 1). The above discussion pertains to structural modeling on the basis of given distributed parameters. The discrete model can be used not only to compute the modes of vibration but also to derive the system response. The problem of deriving the response to given excitations when the system characteristics are known, albeit only approximately, can be regarded as a direct problem.

On occasions, the system parameters are not known and the object is to infer them from measured system response to known excitations. This represents an inverse problem, more commonly known as parameter identification, or parameter estimation. Clearly, in the case of distributed structures, the object is to identify parameter distributions. The implication is that the parameters are distributed nonuniformly, because otherwise the problem would be almost trivial, reducing to the identification of a single number per parameter distribution. But, nonuniform parameter distributions are precisely those preventing closed-form solutions and demanding discrete models. Hence, if the object is to identify nonuniform parameter distributions, then one must expect at the very least the same kind of idiosyncrasies encountered in mere modeling, so that one must proceed with extreme caution.

Commonly used discretization procedures, for modeling as well as for parameter identification, are the classical Rayleigh-Ritz method, the finite element method, the Galerkin method, etc. (Ref. 1). They all end up characterizing the system parameters by means of matrices, such as the mass and stiffness matrices. But, this is the very same process that proved incapable of giving wholly accurate eigensolutions. Indeed, it is typical of the above techniques that less than one half of the computed natural frequencies, and even fewer natural modes, are sufficiently accurate (Ref. 1). Hence, it is unreasonable to expect that a process identifying parameter matrices can yield more accurate results than those obtained by a corresponding process in mere modeling.

There are several factors that call for a cautious approach to parameter identification in distributed structures. In particular, one must interpret and use the results of the identification process

judiciously. The mass and stiffness matrices are not unique for a structure, and in fact they do not even represent physical quantities. Indeed, the dimension of the matrices depends on the number of admissible functions used and the entries of the matrices depend on the nature of the admissible functions. These two factors are in fact related, as a wise choice of admissible functions can give superior results with a smaller number of admissible functions. In particular, one must make sure that the admissible functions are such that all the boundary conditions can be satisfied by finite linear combinations of these functions (Ref. 2). The mass and stiffness matrices themselves have no physical meaning. For distributed structures, the quantities having physical meaning are the mass and stiffness distributions. Perhaps the idea can be best illustrated by using the analogy with the construction of a building, in which the mass and stiffness matrices represent the scaffolding and the mass and stiffness distributions represent the building itself. Hence, an identification process must not have as its goal the identification of the mass and stiffness matrices, but the identification of the mass and stiffness distributions (Ref. 3). At times, the identification of natural frequencies and modes can serve as an intermediate step (Refs. 4 and 5). Another factor affecting the quality of the results is the design of the experiment. In particular, one must ensure that the actuators and sensors are at significant points of the structure and that they are in sufficient number. Of course, noisy actuators and sensors can cause difficulties and even lead to instability in the identification process (Ref. 6).

In modeling, the question arises as to the effect of truncation on the system eigenvalues. Specifically, what happens to the system

eigenvalues when the number of admissible functions is increased (or decreased) in the modeling process? The answer to this question lies in the inclusion principle (Ref. 1), which can be used to verify the accuracy of a model. Because parameter identification in distributed structures depends heavily on modeling, the question can be asked whether an inclusion principle exists in parameter identification.

Practical limitations dictate that control of distributed structures be designed and implemented on the basis of truncated models, whereby only a subset of the modes of the structure are actively controlled. The excitation of the residual (uncontrolled) modes caused by the finite-dimensional controller is known as control spillover and can degrade the system performance. The contamination of the sensor measurements by the residual modes is known as observation spillover, and the combination of control and observation spillover can destablize the system (Ref. 7). An analogous effect exists in the identification problem. Because distributed structures must be modeled by discrete systems, the contamination from the residual (unmodeled) modes leads to identification spillover. Identification spillover can cause errors in the parameter estimates.

This paper stresses the importance of judicious modeling for parameter identification. Distributed-parameter systems can be approximated by a variety of models. The accuracy of these models depends on the type and number of admissible functions used in the discretization process. As shown in the numerical example, the accuracy of the identified parameters depends on the type and number of admissible functions. In parameter identification, the system parameters can be estimated directly using the system response to known

input (Refs. 3 and 6). In this paper, the modal quantities are identified first, and then the system parameters are identified on the basis of the estimated modal quantities (Ref. 4).

Modal identification of distributed structures can be performed using a variational approach (Ref. 5). A ratio of the maximum potential energy to the reference kinetic energy, known as the Rayleigh quotient, can be used to generate the eigenvalues of the model. Based on the inclusion principle, we conclude that, when the order of the model increases, the eigenvalues of the model approach the eigenvalues of the distributed structure monotonically from above. The variational modal identification technique employs the temporal and spatial properties of distributed structures to generate a pseudo-Rayleigh quotient, in which stationary values of the quotient approximate the eigenvalues of the actual system. Moreover, as the order of the identified model is increased, the identified eigenvalues approach the actual eigenvalues monotonically from above. A numerical example demonstrates that the inclusion principle holds for the variational modal identification technique.

Equation of Motion

We consider the case in which the linear motion of the distributed structure is governed by the partial differential equation

$$\mathcal{L}u(P,t) + m(P)\ddot{u}(P,t) = f(P,t) \quad , \quad P \in D \tag{1}$$

where $\mathcal{L}$ is a self-adjoint positive definite differential operator of order 2p, m is the mass density, u is the displacement at the spatial position P at time t, f is the external force density and D is the domain of extension of the system. The displacement u satisfies the

boundary conditions

$$B_i u(P,t) = 0, \quad i = 1,2,\ldots,p \quad , \quad P \quad S \tag{2}$$

where B_i are homogeneous differential operators of maximum order 2p-1 and S is the boundary of D. The boundary conditions are either geometric, in which case the order of B_i is smaller than p, or natural, in which case the order of B_i is smaller than 2p. Functions that are p times differentiable and satisfy the geometric boundary conditions are called admissible functions. Functions that are 2p times differentiable and satisfy all the boundary conditions are called comparison functions.

Associated with Eq. (1), we have the eigenvalue problem

$$\mathcal{L}\phi(P) = \lambda m(P)\phi(P) \tag{3}$$

where $\phi(P)$ are functions satisfying the boundary conditions, Eq. (2). The solution to Eq. (3) consists of a denumerably infinite set of real eigenfunctions $\phi_r(P)$ and associated real positive eigenvalues λ_r $(r = 1,2,\ldots)$. For convenience, we order the eigenvalues so that $\lambda_1 \leq \lambda_2 \leq \ldots$. The eigenfunctions are orthogonal and they can be normalized so as to satisfy the orthonormality conditions

$$\int_D m(P)\phi_r(P)\phi_s(P)dD = \delta_{rs}, \quad \int_D \phi_r(P)\mathcal{L}\phi_s(P)dD = \lambda_r\delta_{rs} \tag{4}$$

where δ_{rs} is the Kronecker delta function. From the expansion theorem (Ref. 1), we can express the displacement u as a linear combination of the eigenfunctions, or

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P)u_r(t) \tag{5}$$

where $u_r(t)$ are modal coordinates. Introducing Eq. (5) into Eq. (1), considering Eqs. (4) and following the usual steps, we obtain the infinite set of second-order ordinary differential modal equations

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t), \quad r = 1,2,... \tag{6}$$

where $\lambda_r = \omega_r^2$, in which ω_r represent the natural frequencies of oscillation, and

$$f_r(t) = \int_D \phi_r(P) f(P,t) dD, \quad r = 1,2,... \tag{7}$$

are modal forces.

In the case of free vibration, Eqs. (6) reduce to

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = 0, \quad r = 1,2,... \tag{8}$$

Equations (8) are independent, so that distinct modes of vibration are uncorrelated, or

$$< u_r(t), u_s(t) > = \lim_{T\to\infty} \frac{1}{T} \int_0^T u_r(t) u_s(t)\, dt = Q_r \delta_{rs}, \quad r,s = 1,2,... \tag{9}$$

where the symbol $< , >$ represents a temporal inner product and Q_r ($r = 1,2,...$) are positive constants. Using the solution of Eqs. (8), it can be shown that

$$< \dot{u}_r(t), \dot{u}_s(t) > = \omega_r^2 Q_r \delta_{rs}, \quad r,s = 1,2,... \tag{10}$$

The modal coordinates $u_r(t)$ can be obtained from the system response by using the modal filters (Ref. 8)

$$u_r(t) = \int_D m(P) \phi_r(P) u(P,t) dD \quad , \quad r = 1,2,... \tag{11}$$

The eigenvalue problem given by Eqs. (2) and (3) can be replaced by a variational problem consisting of determining the stationary values of the Rayleigh quotient

$$R(\psi(P)) = \frac{[\psi(P), \psi(P)]}{\int_D m(P) \psi^2(P) dD} \tag{12}$$

where $\psi(P)$ is a trial function and $[,]$ is an energy inner product (Ref. 1), the latter being obtained through integrations by parts

of $\int_D \psi(P)\mathcal{L}\psi(P)dD$. Note that the energy inner product [,] is symmetric because the stiffness operator $\mathcal{L}$ in Eq. (1) is self-adjoint. Moreover, the energy inner product [u, u] is proportional to the potential energy of the system (Ref. 1). Expressing the trial function in Eq. (12) as a linear combination of the eigenfunctions, or

$$\psi(P) = \sum_{r=1}^{\infty} c_r \phi_r(P) \tag{13}$$

and considering the spatial orthonormality conditions, Eqs. (4), Eq. (12) becomes

$$R(c_1, c_2, \ldots) = \frac{\sum_{r=1}^{\infty} c_r^2 \omega_r^2}{\sum_{r=1}^{\infty} c_r^2} \tag{14}$$

We note that the stationary values of the Rayleigh quotient R are identical to the system eigenvalues and occur every time the trial function $\psi(P)$ is identical to an eigenfunction (Ref. 1).

Closed-form solutions to Eq. (1) exist in a few simple cases only, namely cases in which the parameter distributions are uniform. For more complex structures, an approximate solution must be sought. To this end, the partial differential equation must be discretized in space, so that an approximate solution is obtained by means of a finite set of ordinary differential equations. Commonly used discretization procedures are the classical Rayleigh-Ritz method, the finite element method and the Galerkin method. The difference between the Rayleigh-Ritz and Galerkin method lies in the choice of trial functions, in the sense that Galerkin trial functions must be comparison functions, satisfying all the boundary conditions. If the Rayleigh-Ritz method employs the variational formulation given by Eq. (12), then the trial functions can be merely admissible functions, satisfying the geometric

boundary conditions alone. Of course, the Rayleigh-Ritz method is applicable only to self-adjoint systems, whereas Galerkin's method can be used for non-self-adjoint systems as well (Ref. 1). The finite element method is a variant of the classical Rayleigh-Ritz method, and the trial functions are local admissible functions, as opposed to the trial functions in the classical Rayleigh-Ritz method, which are global admissible functions.

Approximate Methods. The Rayleigh-Ritz Method

Consider the admissible functions $\psi_r(P)$ $(r = 1,2,\ldots,n)$ satisfying the geometric boundary conditions and express the displacement $u(P,t)$ as a linear combination of the admissible functions $\psi_r(P)$ multiplied by time-dependent generalized coordinates $q_r(t)$ as follows:

$$u^{(n)}(P,t) = \sum_{r=1}^{n} \psi_r(P) q_r(t) = \underset{\sim}{\psi}^T(P) \underset{\sim}{q}(t) \tag{15}$$

where $u^{(n)}(P,t)$ is the n^{th}-order approximation of $u(P,t)$, $\underset{\sim}{\psi}(P) = [\psi_1(P)\ \psi_2(P)\ \ldots\ \psi_n(P)]^T$ is an n-vector of admissible functions and $\underset{\sim}{q}(t) = [q_1(t)\ q_2(t)\ \ldots\ q_n(t)]^T$ is an n-vector of generalized coordinates. As n approaches infinity, the Rayleigh-Ritz solution $u^{(n)}(P,t)$ converges to the exact solution $u(P,t)$, provided the trial functions $\psi_r(P)$ $(r = 1,2,\ldots)$ are complete in energy (Ref. 1). Substituting Eq. (15) into Eq. (1), multiplying on the left by $\underset{\sim}{\psi}(P)$ and integrating over the domain D of the structure, we obtain the n-degree-of-freedom discretized model

$$M^{(n)} \ddot{\underset{\sim}{q}}(t) + K^{(n)} \underset{\sim}{q}(t) = \underset{\sim}{F}^{(n)}(t) \tag{16}$$

where

$$M^{(n)} = \int_D m(P) \underset{\sim}{\psi}(P) \underset{\sim}{\psi}^T(P)\, dD, \quad K^{(n)} = [\underset{\sim}{\psi}(P), \underset{\sim}{\psi}^T(P)] \tag{17a,b}$$

are $n \times n$ mass and stiffness matrices, respectively, and

$$\underset{\sim}{F}^{(n)}(t) = \int_D \underset{\sim}{\psi}(P) f(P,t)\, dD \tag{18}$$

is an n-dimensional generalized force vector.

An attractive feature of the Rayleigh-Ritz method is that, by increasing the number of degrees of freedom n, the previously calculated mass and stiffness coefficients do not change, so that one need only calculate an additional row (or column) to obtain the updated mass and stiffness matrices. For an (n + 1)-degree-of-freedom discretized model, the mass and stiffness matrices have the form

$$M^{(n+1)} = \begin{bmatrix} M^{(n)} & \begin{matrix} x \\ x \end{matrix} \\ \begin{matrix} x & x \end{matrix} & x \end{bmatrix}, \quad K^{(n+1)} = \begin{bmatrix} K^{(n)} & \begin{matrix} x \\ x \end{matrix} \\ \begin{matrix} x & x \end{matrix} & x \end{bmatrix} \tag{19a,b}$$

The eigenvalue problems associated with the discrete models are given by

$$\lambda_r^{(n)} M^{(n)} \underset{\sim}{x}_r^{(n)} = K^{(n)} \underset{\sim}{x}_r^{(n)}, \quad r = 1,2,\ldots,n \tag{20a}$$

$$\lambda_r^{(n+1)} M^{(n+1)} \underset{\sim}{x}_r^{(n+1)} = K^{(n+1)} \underset{\sim}{x}_r^{(n+1)}, \quad r = 1,2,\ldots,n+1 \tag{20b}$$

The orthonormality conditions require that the solutions to Eq. (20a) be normalized so that

$$\underset{\sim}{x}_r^{(n)T} M^{(n)} \underset{\sim}{x}_s^{(n)} = \delta_{rs}, \quad \underset{\sim}{x}_r^{(n)T} K^{(n)} \underset{\sim}{x}_s^{(n)} = \lambda_r^{(n)} \delta_{rs} \tag{21a,b}$$

From the inclusion principle (Ref. 1), the relationship between the eigenvalues $\lambda_r^{(n)}$ (r = 1,2,...,n) of the n-degree-of-freedom model and the eigenvalues $\lambda_r^{(n+1)}$ (r = 1,2,...,n + 1) of the (n + 1)-degree-of-freedom model is given by

$$\lambda_1^{(n+1)} \le \lambda_1^{(n)} \le \lambda_2^{(n+1)} \le \lambda_2^{(n)} \le \ldots \le \lambda_n^{(n)} \le \lambda_{n+1}^{(n+1)} \tag{22}$$

As the number of degrees of freedom n in the discretized model is increased, the approximate eigenvalues decrease monotonically and approach the actual eigenvalues of the system asymptotically from above (Ref. 1).

For complex structures, the system parameters, which in the case at hand are the mass and stiffness distributions, are complicated functions of the spatial variables. Because closed-form solutions to Eq. (1) exist in simple cases only, almost always involving uniform parameter distributions, we must obtain an approximate solution using Eq. (16). This implies that the use of Eq. (1) for identification purposes is not practical, so that an exact parameter identification is not feasible. The inverse problem of identifying the unknown parameters represents a compounded version of the direct or modeling problem, where in the latter the parameters are known. We must expect at least the same idiosyncrasies in parameter identification as those encountered in modeling. A suitable choice of the number and type of trial functions used in modeling to obtain Eq. (16) must be determined prior to any parameter identification process.

A Rayleigh-Ritz Type Parameter Expansion

As in the Rayleigh-Ritz method, we expand the parameter distributions in terms of a set of known functions multipled by undetermined coefficients. It is assumed that accurate representations of the mass and stiffness distributions are given by (Ref. 4)

$$m(P) = \sum_{r=1}^{g} \alpha_r m_r(P), \quad k(P) = \sum_{r=1}^{h} \beta_r k_r(P) \tag{23a,b}$$

where $m_r(P)$ and $k_r(P)$ are functions from complete sets and α_r and β_r are undetermined coefficients playing the role of unknown parameters in the identification process. The functions $m_r(P)$ and $k_r(P)$ can be global or local functions. Using Eqs. (17) and (23), it can be shown that the coefficient matrices in Eq. (16) can be written in the form (Ref. 3)

$$M^{(n)} = \sum_{r=1}^{g} \alpha_r M_r^{(n)}, \quad K^{(n)} = \sum_{r=1}^{h} \beta_r K_r^{(n)} \tag{24a,b}$$

where

$$M_r^{(n)} = \int_D m_r(P)\underset{\sim}{\psi}(P)\underset{\sim}{\psi}^T(P)\, dD \quad , \quad K_r = [\underset{\sim}{\psi}(P), \underset{\sim}{\psi}^T(P)]_r \qquad (25a,b)$$

Equations (24) are not new. In fact, Ref. 9 uses submatrix scaling to define distributed parameters where the mass and stiffness matrices are expanded into a linear series of submatrices as given by Eq. (24). One must proceed with caution, however, as Eqs. (15) and (23) represent approximations to the displacement profile and distributed parameters, respectively. Indeed, the discretized model given by Eq. (16) was obtained as an approximate solution to Eq. (1). Moreover, the discretized representation given by Eq. (16) is not unique as the entries in the mass and stiffness matrices depend on the type of admissible functions used in Eqs. (15) and (25). The accuracy of the discretized model depends on the type and number of admissible functions used (Ref. 2). It follows that the accuracy of the estimated parameters in the identification process also depends on the type and number of admissible functions used.

Parameter Identification

In this section, we present an algorithm for identifying parameter distributions in distributed structures. The algorithm represents an iterative procedure in which the parameter distributions are updated using a set of natural frequencies identified from the measured response of the distributed structure. In the next section, we present a method for identifying the natural frequencies from the system response. In the ensuing discussion, we refer to the identified natural frequencies of the distributed structure as the measured natural frequencies. We adopt an incremental approach in which a vector $\underset{\sim}{\omega} = [\omega_1\ \omega_2\ \cdots\ \omega_f]^T$ of f

identified natural frequencies of the actual distributed structure, playing the role of a measurement vector, is obtained and then used to update a parameter vector given by $\underset{\sim}{p} = [\alpha_1\ \alpha_2 \cdots \alpha_g\ \beta_1\ \beta_2 \cdots \beta_h]^T$. The iteration process is based on the incremental relations

$$\Delta\underset{\sim}{\omega} = \underset{\sim}{\omega} - \underset{\sim}{\omega}^{(n)}\ ,\quad \Delta\underset{\sim}{\omega} = \left[\frac{\partial\omega_i}{\partial p_j}\right]\Delta\underset{\sim}{p}\ ,\quad \Delta\underset{\sim}{p} = \underset{\sim}{p} - \underset{\sim}{p}_o \qquad (26a,b,c)$$

To begin the parameter identification, we postulate a parameter vector $\underset{\sim}{p}_o$, compute the corresponding mass and stiffness matrices using Eqs. (24) and (25) and solve for the associated natural frequencies $\underset{\sim}{\omega}^{(n)} = [\sqrt{\lambda_1^{(n)}}\quad \sqrt{\lambda_2^{(n)}} \cdots \sqrt{\lambda_f^{(n)}}]^T$ using Eq. (20a). Then, we use the measurement vector $\underset{\sim}{\omega}$ to compute $\Delta\underset{\sim}{\omega}$ by using Eq. (26a). The Jacobian matrix in Eq. (26b) can be computed by using the orthogonality conditions, Eqs. (21). The entries are

$$\frac{\partial\omega_i}{\partial p_j} = \frac{1}{2\omega_i}\left[\underset{\sim}{x}_i^{(n)T}\frac{\partial K^{(n)}}{\partial p_j}\underset{\sim}{x}_i^{(n)} - \omega_i^2\underset{\sim}{x}_i^{(n)T}\frac{\partial M^{(n)}}{\partial p_j}\underset{\sim}{x}_i^{(n)}\right],$$
$$i = 1,2,\ldots,f\quad j = 1,2,\ldots,g+h \qquad (27)$$

where the eigenvectors $\underset{\sim}{x}_i^{(n)}$ are computed using Eq. (20a) and the entries of the mass and stiffness sensitivity matrices are given by

$$\frac{\partial M^{(n)}}{\partial p_j} = M_j^{(n)},\quad \frac{\partial K^{(n)}}{\partial p_j} = 0,\quad j = 1,2,\ldots,g$$
$$\frac{\partial M^{(n)}}{\partial p_j} = 0,\quad \frac{\partial K^{(n)}}{\partial p_j} = K_j^{(n)},\quad j = g+1,\ g+2,\ \ldots,\ g+h \qquad (28)$$

Then, the increment $\Delta\underset{\sim}{p}$ can be computed by means of Eq. (26b). For the case in which the number f of measured natural frequencies is greater than the number of parameters g + h, a least-squares solution of Eq. (26b) yields

$$\Delta \underline{p} = \left\{ \left[\frac{\partial \omega_i}{\partial p_j} \right]^T \left[\frac{\partial \omega_i}{\partial p_j} \right] \right\}^{-1} \left[\frac{\partial \omega_i}{\partial p_j} \right]^T \Delta \underline{\omega} \tag{29}$$

Finally, from Eq. (26c), we obtained the improved parameter vector

$$\underline{p} = \underline{p}_o + \Delta \underline{p} \tag{30}$$

The procedure is repeated using Eqs. (26a), (27), (29) and (30) until convergence is achieved.

An Inclusion Principle in Parameter Identification

In the preceding section, the estimated parameters were updated using the identified natural frequencies of the distributed structure. The objective is to identify the natural frequencies $\omega_r (r = 1,2,\ldots,f)$ from the free response. To this end, we define a pseudo-Rayleigh quotient suitable for modal identification in the form (Ref. 5)

$$R(\zeta(P)) = \frac{< \dot{w}(t), \dot{w}(t) >}{< w(t), w(t) >} \tag{31}$$

where

$$w(t) = \int_D \zeta(P) u(P,t) dD \tag{32}$$

in which $\zeta(P)$ is an admissible function, i.e., a function satisfying the geometric boundary conditions. We can express the admissible function $\zeta(P)$ as

$$\zeta(P) = \sum_{r=1}^{\infty} a_r m(P) \phi_r(P) \tag{33}$$

where $\phi_r(P)$ are eigenfunctions. Introducing Eqs. (5) and (33) into Eq. (32) and the result into Eq. (31), considering the orthonormality conditions, Eqs. (4), and the temporal correlation properties, Eqs. (9) and (10), we obtain

$$R(a_1,a_2,\ldots) = \frac{\sum_{r=1}^{\infty} a_r^2 Q_r \omega_r^2}{\sum_{r=1}^{\infty} a_r^2 Q_r} \tag{34}$$

Note that Eq. (34) is identical to Eq. (14) with $c_r = a_r\sqrt{\bar{Q}_r}$, so that stationary values of the pseudo-Rayleigh quotient, Eq. (31), are identical to the eigenvalues $\lambda_r = \omega_r^2$ and they occur when $\zeta(P)$ is equal to an eigenfunction multiplied by the mass distribution.

In practice, it is only possible to identify a finite number of natural frequencies and associated modes of vibration. To this end, we express the trial function $\zeta(P)$ as a finite linear combination of the admissible functions $\psi_r(P)$ as follows:

$$\zeta^{(m)}(P) = \sum_{r=1}^{m} v_r \psi_r(P) \tag{35}$$

where $\zeta^{(m)}(P)$ is the m^{th}-order approximation of $\zeta(P)$ and where v_r are undetermined coefficients. Introducing Eqs. (15) and (35) into Eq. (32), we have

$$w(t) = \sum_{r=1}^{m} \sum_{i=1}^{n} v_r \psi_{ri} q_i(t), \quad \psi_{ri} = \int_D \psi_r(P)\psi_i(P)dD \tag{36a,b}$$

Introducing Eqs. (36) into Eq. (31), we obtain

$$R(v_1, v_2, \ldots, v_m) = \frac{\sum_{r=1}^{m} \sum_{s=1}^{m} k_{rs}^{(m)} v_r v_s}{\sum_{r=1}^{m} \sum_{s=1}^{m} m_{rs}^{(m)} v_r v_s} \tag{37}$$

where

$$m_{rs}^{(m)} = m_{sr}^{(m)} = \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ri}\psi_{sj} \langle q_i(t), q_j(t) \rangle, \quad r,s, = 1,2,\ldots,m \tag{38}$$

$$k_{rs}^{(m)} = k_{sr}^{(m)} = \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ri}\psi_{sj} \langle \dot{q}_i(t), \dot{q}_j(t) \rangle, \quad r,s, = 1,2,\ldots,m$$

Determining the stationary values of Rayleigh's quotient, Eq. (37), can be replaced by the eigenvalue problem (Ref. 1)

$$\lambda^{(m)} M^{(m)} \underset{\sim}{v}^{(m)} = K^{(m)} \underset{\sim}{v}^{(m)} \tag{39}$$

where the solution to Eq. (39) represents the m^{th}-order approximation to the stationary values of the pseudo-Rayleigh quotient, Eq. (31), and

where $M^{(m)}$, $K^{(m)}$ and $\underset{\sim}{v}^{(m)}$ have corresponding entries $[m_{rs}^{(m)}]$, $[k_{rs}^{(m)}]$ and $v_r^{(m)}$, respectively. The m^{th}-order approximate eigenfunctions can be computed from the eigenvectors $\underset{\sim}{v}^{(m)}$ using

$$\underset{\sim}{\phi}_r^{(m)}(P) = \underset{\sim}{u}^T(P)\underset{\sim}{v}_r^{(m)} \tag{40}$$

The solution to Eqs. (39) and (40) represent the identified mth-order approximation to the eigenvalues and eigenfunctions of the actual system, in the same way as the solution to Eq. (20a) represents the nth-order approximation to Eq. (3).

When m = n in Eqs. (38), the orders of the model and identified model are equivalent, so that in this case identification spillover does not occur. For the model to be an accurate representation of a distributed structure, the number of degrees of freedom n can be quite large, so that in practice the number m of identified natural frequencies and mode shapes can be much smaller than n. In the case in which m < n, identification spillover is present. Note that, by increasing m in Eqs. (38), previously calculated entries in $M^{(m)}$ and $K^{(m)}$ do not change. Hence, as in modeling (Ref. 1), the identified eigenvalues satisfy the inclusion principle as given by inequalities (22).

Numerical Example

As an illustration, we consider a rod in axial vibration with a spring of spring constant k attached to the free end as shown in Fig. 1. For this case, the stiffness operator $\mathcal{L}$ in Eq. (1) has order 2 (p = 1) and is given by

$$\mathcal{L} = \frac{\partial}{\partial x}\left[EA(x)\,\frac{\partial}{\partial x}\right] \tag{41}$$

where EA(x) is the axial stiffness. The boundary conditions are given

by

$$B_1 = 1 \quad \text{at } x = 0, \quad B_1 = EA \frac{\partial}{\partial x} + k \quad \text{at } x = L \tag{42}$$

Moreover, the energy inner product in Eq. (12) has the form

$$[u, u] = \int_0^L EA \left(\frac{\partial u}{\partial x}\right)^2 dx + ku^2(L,t) \tag{43}$$

The mass and stiffness distributions for the actual model are given by

$$m(x) = \frac{6m}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right], \quad EA(x) = \frac{6EA}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right] \tag{44a,b}$$

As an illustration, we consider the case where the distributions are known with unknown scaling such that $g = h = 1$ and that $m_1(x) = \frac{6}{5}\left[1 - \frac{1}{2}\left(\frac{x^2}{L}\right)\right]$, $k_1(x) = \frac{6}{5}\left[1 - \frac{1}{2}\left(\frac{x^2}{L}\right)\right]$ in Eqs. (23). Hence, the problem reduces to identifying three parameters, namely, m, EA and k. Moreover, because free vibration data is used, to identify the parameters uniquely, one of the parameters must be known. For convenience, we chose m = 1 kg/m.

The first objective is to show that accurate modeling has a direct influence on the accuracy of the identification results. The system does not have a closed-form solution so that we must resort to spatial discretization. To this end, we consider two sets of trial functions $\psi_r(P)$ $(r = 1,2,\ldots,n)$ to be used in Eq. (15). The first set consists of the admissible functions

$$\psi_r(x) = \sin \frac{(2r - 1)\pi x}{2L}, \quad r = 1,2,\ldots,n \tag{45}$$

We note that these functions satisfy the geometric boundary condition u = 0 at x = 0. They represent the eigenfunctions to a related problem, i.e., they are the eigenfunctions of a uniform rod with no spring at the free end. The second set of trial functions consists of the functions

$$\xi_r(x) = \sin \frac{r\pi x}{2L}, \quad r = 1,2,\ldots,n \tag{46}$$

These functions are also admissible functions and they represent quasi-comparison functions for this example (Ref. 2). Note that a finite linear combination of the quasi-comparison functions can satisfy the natural boundary condition (i.e., $B_1 = EA \frac{\partial}{\partial x} + k$ at $x = L$), whereas the first set of admissible functions, although complete in energy, can only satisfy the natural boundary condition when n approaches infinity.

To refine the system parameters EA and k, we used the first three natural frequencies from the actual model so that f = 3 in Eq. (28). As an initially postulated model, we used the parameters EA = 1.5 N and k = 1.5 N/m, while the actual model parameters are EA = 1 N and k = 1 N/m. To check the algorithm, several other cases involving different initially postulated parameters were tested. The results are not presented here, but for reasonable starting values the identification results were insensitive to the initially postulated parameters and the algorithm converged within five iterations. In addition to using ordinary admissible functions and quasi-comparison functions for modeling, different numbers of degrees of freedom were used. Table 1 displays the results. As expected, the identified parameters improve as the number n of degrees of freedom increases. Indeed, as the number n of admissible functions increases, the model becomes more representative of the actual system, resulting in improved identified parameters. The model with the admissible functions can predict the value for EA very well, but the identification of the parameter k is rather poor because a finite number of admissible functions cannot satisfy the natural boundary condition. On the other hand, the parameter identification based on the model using quasi-comparison functions is extremely good for $n > 4$, as both parameters are identified exactly.

The second objective is to illustrate the inclusion principle in parameter identification and demonstrate the effects of identification spillover. To this end, we consider the rod in axial vibration described previously. To identify the natural frequencies from the free response, it was necessary to simulate the response of the actual system. The model for this purpose was obtained by the Rayleigh-Ritz method using n = 8 admissible functions in Eq. (15). The admissible functions used in the discretization were the quasi-comparison functions described by Eq. (46). Different order models were identified using the free response, where m = 4,5,6 and 7 degrees of freedom were considered in Eqs. (39). With the quasi-comparison functions as admissible functions, Eq. (36b) yields

$$\psi_{ri} = \begin{cases} \frac{L}{2}, & r = i \\ L\left[\frac{\sin (r - i)\pi}{(r - i)\pi} - \frac{\sin (r + i)\pi}{(r + i)\pi}\right], & r \neq i \end{cases} \tag{47}$$

We note that, to compute $q_i(t)$ and $\dot{q}_i(t)$ $(i = 1,2,\ldots,8)$ in Eq. (38), the model was excited using an initial impulse of magnitude 1 N·s acting at x = L. To identify the natural frequencies, the solution of the eigenvalue problem given by Eq. (39) was obtained and the results are presented in Table 2. Note that identification spillover has the effect of degrading the identified results. Indeed, the spillover from the unmodeled degrees of freedom has the effect of increasing the identified natural frequencies. Moreover, as the order of the identified model increases, the estimated natural frequencies decrease monotonically and approach the actual natural frequencies asymptotically from above. It is also obvious from Table 2 that the computed eigenvalues satisfy the inclusion principle, as stated by inequalities (22).

Conclusions

Parameter identification in distributed structures is ordinarily based on discretized models defined by mass and stiffness matrices. Mass and stiffness matrices, however, have no physical meaning for distributed structures, whereas mass and stiffness distributions do. Indeed, the mass and stiffness distributions are the quantities describing the physical characteristics of the structure. Moreover, mass and stiffness matrices are not unique for a given structure and their entries and dimensions depend on the number and type of trial functions used in the modeling process. Hence, the object of a parameter identification technique should be to identify physical properties and not mass and stiffness matrices.

Parameter identification in structures represents a compounded version of the modeling problem in structures. Indeed, as the numerical example indicates, the choice of the model can greatly affect the results of the parameter identification process. The choice of the type and number of admissible functions is at least as important in parameter identification as it is in the problem of modeling distributed structures. A wise choice of the type and number of the admissible functions, can improve the results of the parameter identification process.

Practical limitations dictate that the control of distributed-parameter systems be designed on the basis of finite-order models. The excitation of the unmodeled modes by finite-dimensional controller can degrade the system performance. As in the control problem, identification spillover exists and it has the effect of degrading the estimates of the parameters. It is shown that an inclusion principle

exists in the identification of distributed-parameter structures, so that the identified natural frequencies approach the actual natural frequencies monotonically from above.

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Table 1. Identified Parameters Using Admissible Functions (AF) and Quasi-Comparison Functions (QCF) for Different Degrees of Freedom.

	AF			QCF		
n	EA	k	% RMS ERROR	EA	k	% RMS ERROR
4	1.0016	0.91705	5.87	0.98809	1.0564	4.08
5	1.0010	0.93335	4.71	0.99996	0.99977	0.02
6	1.0008	0.94474	3.91	0.99996	0.99984	0.01
7	1.0006	0.95286	3.33	0.99999	0.99987	0.01
8	1.0005	0.95208	2.89	1.00000	1.00000	0.00

Table 2 Identified Natural Frequencies for Different Degrees of Freedom.

Identified Natural Frequencies (rad/s)				Natural Frequencies of Discrete Model (rad/s)
m = 4	m = 5	m = 6	m = 7	
2.388	2.250	2.249	2.228	2.210
15.083	5.225	5.223	5.133	5.083
21.886	17.165	8.822	8.173	8.087
57.160	23.860	17.198	12.860	11.148
	63.161	23.918	17.308	14.237
		63.351	24.155	17.535
			64.156	24.388
				64.946

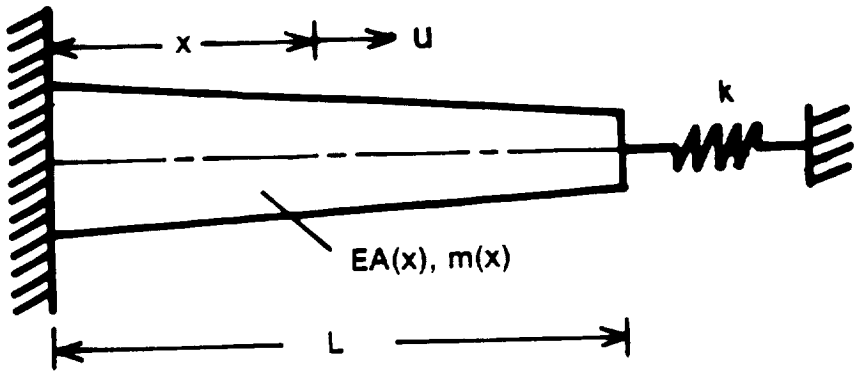

Figure 1. Rod in Axial Vibration with a Spring Attached to the Free End